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LETTER TO THE EDITOR

Band to band hopping in one-dimensional maps†

Scott J Shenker‡ and Leo P Kadanoff

The James Franck Institute and The Department of Physics, The University of Chicago, Chicago, Illinois 60637, USA

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Abstract. One-dimensional maps have chaotic bands which may merge at critical values of the control parameter, a. Just after the merger, one can define a band-to-band hopping time, which diverges as $|a-a_c|^{-1/z}$ for a map with a zth-order maximum. Numerical data are presented to support this conclusion.

It is now widely believed that one-dimensional recursion relations

$$x_{n+1}=f_a(x_n) \tag{1}$$

where f_a has a single maximum provide important clues as to the nature of the onset of chaos in dissipative dynamical systems. Consequently, there has been much effort devoted to describing the properties of these maps, and investigating their implications for real systems. Most of the work has focused on the U-sequence of stable periodic cycles (Metropolis *et al* 1973) that arise as the parameter a is varied. The sequence of period doubling bifurcations, and its universal scaling behaviour, has received particularly intense scrutiny (Feigenbaum 1978, 1979a, Coullet and Tresser 1978a, b, Derrida *et al* 1979). The relevance of this work for real systems has been strikingly demonstrated by recent experiments on stressed fluids which have revealed a similar subharmonic bifurcation structure (Libchaber and Maurer 1979, Gollub *et al* 1980; see also Feigenbaum 1979b, 1980).

The asymptotic nature of the recursion relation changes drastically as we vary a through a_∞, the accumulation point of the first infinite sequence of period doubling bifurcations. The envelope of the Lyapunov exponent becomes non-zero for $a>a_\infty$, and we thus expect chaotic behaviour (Huberman and Rudnick 1980, Chang and Wright 1980). It is important to note that throughout the $a>a_\infty$ regime there are an infinite number of stable cycles (the remainder of the U-sequence). However, available evidence indicates that the set of a's for which there are no stable cycles has positive measure, and it is on this set that the Lyapunov exponent will be positive (Lorenz 1979, Collet and Eckman 1980).

While the $a>a_\infty$ regime may be chaotic in some sense, recent papers have focused on the residual order contained in the band structure (Lorenz 1980, Huberman and Rudnick 1980, Chang and Wright 1980, Coullet and Tresser 1980). The band structure is essentially a mirror image of the period doubling bifurcation structure, except that the periodic cycling is now between bands rather than distinct points. While there is no

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experimental evidence for the relevance of this phenomenon for real systems, Lorenz (1980) and Crutchfield *et al* (1980) have obtained evidence indicating that some of the sharp spectral lines for certain differential equations are due to the 'noisy periodicity' of the band structure. In this Letter, we investigate how the noisy periodicity breaks down when the bands begin to merge.

Consider the family of one-dimensional recursion relations

$$x_{n+1} = f_a(x_n) = 1 - a|x_n|^z \tag{2}$$

defined for $x_n \in [-1, 1]$, $a \in [0, 2]$ and an arbitrary power $z > 0$. For a particular value of a, there is an m-band structure if m is the greatest integer for which there exist m finite disjoint intervals I_i such that

$$\begin{aligned} &(1)\ 0 \in I_0 \\ &(2)\ f(I_i) = I_{i+1} \qquad 0 \leqslant i \leqslant m-2 \\ &(3)\ f(I_{m-1}) = I_0. \end{aligned} \tag{3}$$

If we let a_2 denote the value of a for which $f^3_{a_2}(0) = f^4_{a_2}(0)$ then, for $1 < a < a_2$, the two intervals I_0, I_1 defined below are disjoint and map into each other under f:

$$I_0(a) = [f^4_a(0), f^2_a(0)] \qquad I_1(a) = [f_a(0), f^3_a(0)]. \tag{4}$$

We have at least a two-band structure; i.e. if we take any initial point $x_0 \in I$, and apply f repeatedly, the resulting iterates will alternate between I_0 and I_1. When $a > a_2$, the regions are no longer disjoint and the two bands have merged into a single band (see figure 1). Now, the alternating pattern of x_i will eventually break down. However, the average number of iterations $\bar{n}$ needed for this to happen becomes very large for a close to a_2; in fact, $n \propto (a - a_2)^{-1/z}$. To study this phenomenon in detail it is more convenient to concentrate on the function $f^2_a(x)$. If we are in a two-band regime, f^2_a maps I_0 onto

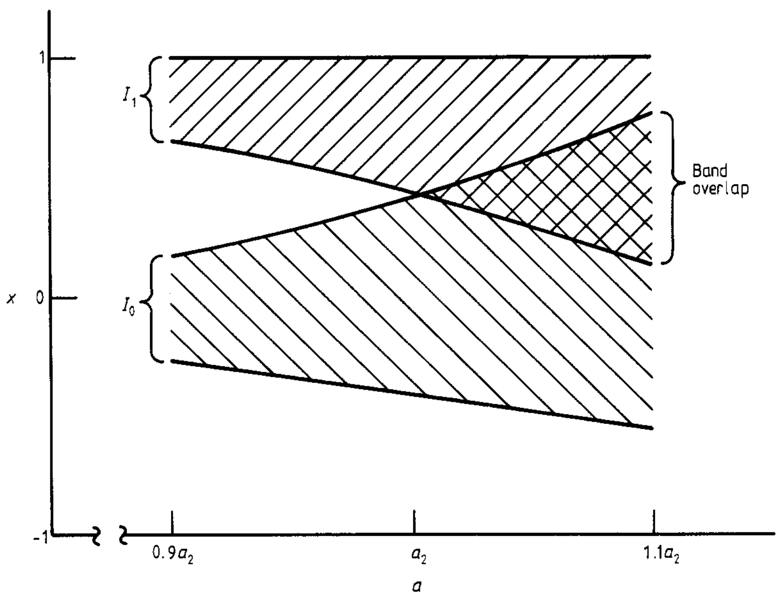

Figure 1. Overlapping of intervals I_0 and I_1.

itself so that all double iterates x_{2i} of x_0 will remain trapped in I_0. When the bands are merged, the iterates will eventually hop out of I_0 into I_1, and it is this band-to-band hopping process that interests us.

Yorke and Yorke (1979) have made a detailed study of a similar problem, that of metastable chaos. They show that for an invariant distribution of initial conditions x_0 in I_0, the function $\phi(n)$ giving the fraction of points which hop out of I_0 on the nth iteration is an exponential

$$\phi(n) = (1/\bar{n})\,\mathrm{e}^{-n/\bar{n}} \tag{5}$$

and that the average hopping time $\bar{n}$ diverges as

$$\bar{n} \propto \varepsilon^{-1/z}. \tag{6}$$

where $\varepsilon = a - a_2$.

This theory was tested numerically for three values of z: $z = 4$, $z = 2$, $z = \frac{10}{9}$. For a particular value of ε, the average diffusion time was found by averaging the diffusion times of 200 initial points distributed evenly over I_0. Data were collected for more than 30 values of ε for each value of z, and then fitted to a curve $\bar{n}(\varepsilon) = A\varepsilon^{-\gamma}$. The results for the coefficients A and γ are listed in table 1.

This theory has interesting implications for correlation functions of the form

$$F(n) = \langle x f^n(x)\rangle - \langle x\rangle^2 \tag{7}$$

where the brackets denote an average over an invariant distribution. For $a < a_\infty$ (or any

Table 1. Numerical results for the coefficients A and γ in the relation $\bar{n} = Ae^{-\gamma}$.

	A	γ	Theoretical prediction for γ
$z = 4$	2.34 ± 0.19	0.261 ± 0.013	0.25
$z = 2$	1.55 ± 0.21	0.514 ± 0.015	0.5
$z = \frac{10}{9}$	0.86 ± 0.11	0.898 ± 0.021	0.9

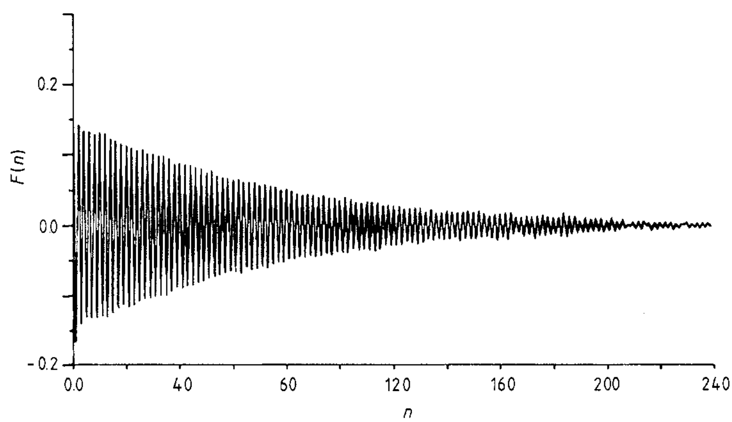

Figure 2. Correlation function $F(n)$ computed for $\varepsilon = 4 \times 10^{-4}$ ($\bar{n} \sim 85$). n is the number of iterations.

a for which a stable cycle exists), $F(n)$ remains non-zero as $n \to \infty$. Also, when $a_\infty < a < a_2$, $F(n)$ has non-trivial long-time behaviour due to the alternating nature of the band structure (Grossman and Thomae 1977). However, for a slightly above a_2, the exponential decay in the band structure produces an exponential decay in the correlation function:

$$F(n) \propto (-1)^n \, \mathrm{e}^{-n/\bar{n}} \tag{8}$$

(see figure 2). Thus, the Fourier transform $\tilde{F}(\omega)$ of $F(n)$ will have a sharp peak at $\omega = \pi$ for $a < a_2$, and this line will broaden into a Lorentzian shaped peak with linewidth proportional to $\varepsilon^{1/z}$ for $\varepsilon = a - a_2 > 0$.

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